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# Observer-Based Robust-$H_\infty$ Control Laws for Uncertain Linear Systems

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**Based on the algebraic Riccati equation approach, this paper presents a simple, flexible method for designing full-order observer-based robust-$H_\infty$ control laws for linear systems with structured parameter uncertainty. The observer-based robust-$H_\infty$ output-feedback control law, obtained by solving three augmented algebraic Riccati equations, provides both robust stability and disturbance attenuation with $H_\infty$-norm bound for the closed-loop uncertain linear system. Several tuning parameters are embedded into the augmented algebraic Riccati equations so that flexibility in finding the symmetric positive-definite solutions (and hence the robust-$H_\infty$ control laws) is significantly increased. A benchmark problem associated with a mass-spring system, which approximates the dynamics of a flexible structure, is used to illustrate the design methodologies, and simulation results are presented.**

## Introduction

THE problems of robust stabilization of uncertain linear systems have been studied by many researchers. The algebraic Riccati equation (ARE) approach to the stabilization of the systems with structured parameter uncertainty has been developed by Petersen and Hollot,[1] Petersen,[2] Schmitendorf,[3] Jabbari and Schmitendorf,[4] Khargonekar et al.,[5] and Wang et al.[6] These approaches have generally utilized the concept that a given ARE-based control law guarantees the existence of a quadratic Lyapunov function (and hence stability) for a closed-loop uncertain linear system. Also, other recent research attention, (e.g., Petersen,[7] Khargonekar et al.,[8] Glover and Doyle,[9] Bernstein and Haddad,[10] Doyle et al.,[11] and Scherer[12]) has shown that ARE-based robust controllers are able not only to stabilize the linear systems with no uncertain parameters but also reduce the effect of disturbances on the controlled output to a prespecified level.

Moreover, Madiwale et al.[13] and Veillette et al.[14] have proposed alternative ARE-based robust controllers that provide both robust stability and disturbance attenuation with $H_\infty$-norm bounds for the systems with structured parameter uncertainty. In Ref. 13 their design method involves solving, in general, four coupled modified Riccati/Lyapunov equations, and the designed full-order and/or reduced-order robust controllers guarantee robust stability, robust ($H_2$) performance, and $H_\infty$ disturbance attenuation for uncertain linear systems. In Ref. 14, in order to obtain full-order robust controllers for both robust stability and disturbance rejection with $H_\infty$-norm bounds, their design method embedded the information of structured system uncertainty into the AREs that were used for nominal $H_\infty$ disturbance-attenuation design.

In this paper we present a simple, flexible method based on the ARE approach for determining full-order observer-based robust-$H_\infty$ output-feedback dynamic control laws for the systems with structured parameter uncertainty. The developed state- and output-feedback control laws provide both robust stability and $H_\infty$ disturbance attenuation for closed-loop uncertain linear systems. The design procedure is described in the following. First, we determine a robust-$H_\infty$ state-feedback control law by solving the first augmented ARE, which accounts for both structured system uncertainty and $H_\infty$ disturbance attenuation. Second, based on a dual concept, a full-order robust-$H_\infty$ observer is obtained by solving the second augmented ARE, which is dual to the one used in designing the robust-$H_\infty$ state-feedback control law. The structure of our observer is different from that developed in Ref. 11, which considers the estimation of the worst disturbance. Third, when the third augmented ARE has a symmetric positive-definite (SPD) solution, the resulting observer-based dynamic control law guarantees both robust stability and $H_\infty$ disturbance attenuation for a closed-loop uncertain linear system. Several tuning parameters are embedded into the augmented AREs to enhance flexibility in finding the SPD solutions and hence the robust-$H_\infty$ control laws.

A benchmark problem associated with a mass-spring system, which approximates the dynamics of a flexible structure,[15-18] is used to illustrate the design methodologies, and simulation results are included.

## Notation and Problem Formulation

Throughout this paper all matrices and vectors are considered to be real and of appropriate dimensions. Also, we denote the following:

$\sigma_{\max}(M)$ = maximum singular value of a matrix $M$
$\sigma_{\min}(M)$ = minimum singular value of a matrix $M$
$\|M\|$ = matrix norm, $\|M\| \triangleq \sigma_{\max}(M)$
$I$ = identity matrix of appropriate dimension
$0$ = null matrix of appropriate dimension
$M>(\geq)0$ = matrix $M$ is symmetric positive (semi-) definite
$M<(\leq)0$ = matrix $M$ is symmetric negative (semi-) definite
$P>(\geq)Q$ = means $P-Q>(\geq)0$
$P<(\leq)Q$ = means $P-Q<(\leq)0$

Consider the uncertain linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + Dw(t) \tag{1a}$$

$$y(t) = Cx(t) + Sv(t) \tag{1b}$$

$$z(t) = \begin{bmatrix} C_1 x(t) \\ D_1 u(t) \end{bmatrix} \tag{1c}$$

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where $x(t)$ is the state, $u(t)$ is the control input, $y(t)$ is the measured output, $w(t)$ and $v(t)$ are disturbances, and $z(t)$ is the controlled output. The unknown but bounded structured uncertainty is described by

$$\Delta A = \sum_{i=1}^{\ell} a_i A_i \tag{2a}$$

where $a_i$ are uncertain parameters and $A_i$ are known constant matrices. Without loss of generality, we assumed that

$$|a_i| \le 1, \qquad i = 1, \ldots, \ell \tag{2b}$$

Note that the uncertainty matrix $\Delta A$ can be time varying. Applying the singular value decomposition (SVD) technique[19] to $A_i$, we can decompose each $\mathbf{A}_i$ as

$$A_i = T_i U_i^T, \qquad i = 1, \ldots, \ell \tag{2c}$$

where $T_i$ and $U_i$ are weighted unitary matrices. We assume that the nominal system trio $(A, B, C)$ is controllable and observable. Let a scalar $\delta > 0$ be the given disturbance-attenuation constant. The problem is to design an output-feedback dynamic control law of the following form:

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{3a}$$

$$u(t) = C_c x_c(t) \tag{3b}$$

such that the closed-loop system in Eqs. (1) is stable and the $H_\infty$ norm of the closed-loop transfer function matrix from the disturbance input $\hat{w}(t) \triangleq [w^T(t), v^T(t)]^T$ to the controlled output $z(t)$ in Eqs. (1) is less than $\delta$ for all $\Delta A$ in Eqs. (2).

When $\Delta A = 0$, $D_1^T D_1 = I$, $SS^T = I$, and the trio $(A, D, C_1)$ is controllable and observable, a necessary and sufficient condition for the existence of a dynamic controller that stabilizes the nominal system in Eqs. (1) and achieves the $H_\infty$-norm bound $\delta$, is given by Doyle et al.[11] as follows:

a) There exists a matrix $P_1 > 0$ satisfying the following Riccati equation:

$$A^T P_1 + P_1 A - P_1\left(BB^T - \frac{1}{\delta^2} DD^T\right)P_1 + C_1^T C_1 = 0 \tag{4}$$

and the resulting matrix $A - [BB^T - (1/\delta^2)DD^T]P_1$ is stable.

b) There exists a matrix $P_2 > 0$ satisfying the following Riccati equation:

$$AP_2 + P_2 A^T - P_2\left(C^T C - \frac{1}{\delta^2} C_1^T C_1\right)P_2 + DD^T = 0 \tag{5}$$

and the resulting matrix $A - P_2[C^T C - (1/\delta^2)C_1^T C_1]$ is stable.

c) The matrix $[I - (1/\delta^2)P_2 P_1]^{-1} P_2$ is symmetric positive definite.

Suppose that there exist $P_1 > 0$ and $P_2 > 0$ satisfying the conditions (a–c); then an observer-based output-feedback dynamic control law is obtained[11] as

$$\dot{x}_c(t) = \left(\mathrm{A} + \frac{1}{\delta^2} DD^T P_1\right) x_c(t) + Bu(t) - \left(I - \frac{1}{\delta^2} P_2 P_1\right)^{-1} P_2 C^T\left[Cx_c(t) - y(t)\right] \tag{6a}$$

$$u(t) = -B^T P_1 x_c(t) \tag{6b}$$

where the term $(1/\delta^2)D^T P_1 x_c(t)$ in Eq. (6a) is interpreted as the estimation of the worst disturbance, i.e., $w(t) = (1/\delta^2)D^T P_1 x(t)$ (Ref. 11).

In this paper we design an alternative observer-based output-feedback dynamic control law of the following form:

$$\dot{x}_c(t) = Ax_c(t) + Bu(t) + K_o\left[Cx_c(t) - y(t)\right] \tag{7a}$$

$$u(t) = K_c x_c(t) \tag{7b}$$

which does not assume the presence of the worst disturbance in the observer design for both robust stabilization and $H_\infty$ disturbance attenuation of the uncertain linear system in Eqs. (1). Applying the dynamic control law in Eqs. (7) to the uncertain system in Eqs. (1) gives the following closed-loop system:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A + \Delta A & BK_c \\ -K_o C & A + BK_c + K_o C \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & -K_o S \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \tag{8a}$$

$$z(t) = \begin{bmatrix} C_1 & 0 \\ 0 & D_1 K_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \tag{8b}$$

By introducing the observer error $e(t) = x(t) - x_c(t)$, we can transform the system in Eqs. (8) to the following augmented system:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + \Delta A + BK_c & -BK_c \\ \Delta A & A + K_o C \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} D & 0 \\ D & K_o S \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \tag{9a}$$

$$z(t) = \begin{bmatrix} C_1 & 0 \\ D_1 K_c & -D_1 K_c \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{9b}$$

Now the problem is reduced to designing a state-feedback gain $K_c$ and an observer gain $K_o$ in Eqs. (7) such that the augmented system in Eqs. (9) is stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t) = [w^T(t), v^T(t)]^T$ to $z(t)$ in Eqs. (9) is less than some prespecified value for all $\Delta A$ in Eqs. (2).

In the following development we utilize the following matrix inequality:

$$\xi XX^H + \frac{1}{\xi} YY^H \pm (XY^H + YX^H) \ge 0 \tag{10}$$

where $X$ and $Y$ are any two appropriately dimensioned matrices, and $\xi$ is a positive scalar. The following lemma will be utilized in the sequel.

*Lemma 1.*[14] Let $\bar{A}$, $\bar{D}$, and $\bar{E}$ be matrices of appropriate dimensions. For a given positive scalar $\delta$, if there exist a SPD matrix $P$ and a positive scalar $\xi$ such that

$$\bar{A}^T P + P\bar{A} + \frac{\xi}{\delta} P\bar{D}\bar{D}^T P + \frac{1}{\xi\delta} \bar{E}^T \bar{E} < 0 \tag{11}$$

then $\bar{A}$ is a stability matrix, and $\bar{H}(s) = \bar{E}(sI - \bar{A})^{-1}\bar{D}$ satisfies

$$\|\bar{H}(s)\|_\infty < \delta \tag{12}$$

■

## Robust State-Feedback Control

If the state of the system in Eqs. (1) is available for measurement and a state-feedback control law is utilized, then the disturbance $v(t)$ in Eq. (1b) has no effect on the controlled output $z(t)$ of the closed-loop system. Let the state-feedback control law be given by

$$u(t) = K_c x(t) \tag{13}$$

The closed-loop system in Eqs. (1) becomes

$$\dot{x}(t) = (A + \Delta A + BK_c)x(t) + Dw(t) \tag{14a}$$

$$z(t) = \begin{bmatrix} C_1 \\ D_1 K_c \end{bmatrix} x(t) \tag{14b}$$

Note that, when $e(t)=0$, the closed-loop system in Eqs. (9) reduces to

$$\dot{x}(t) = (A + \Delta A + BK_c)x(t) + [D, 0]\hat{w}(t) \tag{15a}$$

$$z(t) = \begin{bmatrix} C_1 \\ D_1 K_c \end{bmatrix} x(t) \tag{15b}$$

which is identical to the one in Eqs. (14).

The following theorem is developed for finding a robust-$H_\infty$ state-feedback gain $K_c$ such that the closed-loop system in Eqs. (15) is stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t)$ to $z(t)$ in Eqs. (15) is less than some prespecified value for all $\Delta A$ in Eqs. (2).

*Theorem 1.* Consider the closed-loop system in Eqs. (15) with the structured uncertainty described in Eqs. (2). Let $\delta > 0$ be the given disturbance-attenuation constant. Suppose that there exist positive scalars $\xi_i > 0$, $i = 1, \ldots, \ell$ [where $\ell$ is the number of uncertain parameters as in Eqs. (2)] and $\infty > \hat{\xi} > \sigma^2_{\max}(D_1)/4\delta$, and a matrix $Q_c > 0$ such that the Riccati equation

$$A^T P_c + P_c A + \sum_{i=1}^{\ell} \left( \xi_i P_c T_i T_i^T P_c + \frac{1}{\xi_i} U_i U_i^T \right) + \hat{\xi}\delta\, P_c D D^T P_c + \frac{1}{\hat{\xi}\delta} C_1^T C_1 - P_c B \left( I - \frac{1}{4\hat{\xi}\delta} D_1^T D_1 \right) B^T P_c + Q_c = 0 \tag{16}$$

has a SPD solution $P_c$, where $T_i$ and $U_i$, $i = 1, \ldots, \ell$ are defined in Eq. (2c). Then, the closed-loop system in Eqs. (15) is stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t)$ to $z(t)$ in Eqs. (15) is less than $\delta$ for all $\Delta A$ in Eqs. (2) with

$$K_c = -\gamma_c B^T P_c \tag{17a}$$

where $\gamma_c$ satisfies either

$$\frac{2\hat{\xi}\delta}{\sigma^2_{\max}(D_1)} - \frac{1}{2} \geq \gamma_c \geq \frac{1}{2} \quad \text{or} \quad \frac{1}{2} \geq \gamma_c \geq \frac{2\hat{\xi}\delta}{\sigma^2_{\min}(D_1)} - \frac{1}{2} > 0 \tag{17b}$$

*Proof.* The basis for the proof is similar to that in Ref. 6 and that in Ref. 13, but it is included here for completeness and clarity. Suppose that $P_c > 0$ satisfies the Riccati equation in Eqs. (16). To show that the system in Eqs. (15) with $K_c$ given by Eqs. (17) is stable and satisfies the $H_\infty$-norm bound $\delta$ for all $\Delta A$ in Eqs. (2), by Lemma 1, it suffices to show that

$$\hat{Q}_c \triangleq -(A + \Delta A + BK_c)^T P_c - P_c(A + \Delta A + BK_c) - \frac{\hat{\xi}}{\delta} P_c D D^T P_c - \frac{1}{\hat{\xi}\delta}(C_1^T C_1 + K_c^T D_1^T D_1 K_c)$$

is SPD for all $\Delta A$ in Eqs. (2). From Eqs. (16), it follows that

$$\hat{Q}_c = \sum_{i=1}^{\ell} \left( \xi_i P_c T_i T_i^T P_c + \frac{1}{\xi_i} U_i U_i^T \right) - \Delta A^T P_c - P_c \Delta A + P_c B \left[ (2\gamma_c - 1)I + \frac{1 - 4\gamma_c^2}{4\hat{\xi}\delta} D_1^T D_1 \right] B^T P_c + Q_c$$

Also, using the assumption in Eqs. (2b) and the matrix inequality in Eq. (10), we obtain the following inequality:

$$\sum_{i=1}^{\ell} \left( \xi_i P_c T_i T_i^T P_c + \frac{1}{\xi_i} U_i U_i^T \right) - \Delta A^T P_c - P_c \Delta A \geq \sum_{i=1}^{\ell} |a_i| \left[ \xi_i P_c T_i T_i^T P_c + \frac{1}{\xi_i} U_i U_i^T \pm (P_c T_i U_i^T + U_i T_i^T P_c) \right] \geq 0$$

Now, if $\gamma_c$ satisfies Eq. (17b), it is easy to see that

$$(2\gamma_c - 1)I + \frac{1 - 4\gamma_c^2}{4\hat{\xi}\delta} D_1^T D_1 = (2\gamma_c - 1)\left( I - \frac{2\gamma_c + 1}{4\hat{\xi}\delta} D_1^T D_1 \right) \geq 0$$

As a result, $\hat{Q}_c \geq Q_c > 0$, which completes the proof. ■

*Remark 1.* The Riccati equation in Eq. (16) is constructed to account for both structured uncertainty in Eqs. (2) and $H_\infty$ disturbance attenuation $\delta$. If there is no system uncertainty (i.e., $\Delta A = 0$) and disturbance attenuation is not required (i.e., $\delta \to \infty$), this *augmented* Riccati equation in Eq. (16) reduces to an *ordinary* Riccati equation that arises in the linear quadratic regulator problem.[20] ■

*Remark 2.* It should be noted that a tuning parameter $\gamma_c$ is introduced in Eqs. (17) to give an explicit bound for which $K_c$ is allowed to vary without affecting robust stability and disturbance attenuation of the closed-loop system. In other words, Eqs. (17) can be interpreted as a constraint of the gain margin for the closed-loop uncertain system (15), which has both robust stability and disturbance attenuation. ■

*Remark 3.* The introduction of the weighting matrix $Q_c$, the disturbance-attenuation constant $\delta$, the tuning parameters $\xi_i$, $i = 1, \ldots, \ell$, and $\hat{\xi}$ in Eq. (16) enables the proposed approach to be more flexible in obtaining the robust-$H_\infty$ state-feedback gain $K_c$. In other words, the introduced parameters can be appropriately tuned to select a suitable compromise among system performance, robust stability, and disturbance attenuation. When solving for $K_c$, each tuning parameter $\xi_i$ can be individually and successively decreased until Eq. (16) has a SPD solution. In our experience, when a robust control law exists for a given uncertain linear system, we are able to find the robust control law without any numerical problems. ■

## Robust Output-Feedback Control

When the state is not available for state-feedback design, the design problem reduces to finding an observer-based output-feedback dynamic control law in Eqs. (7) for both robust stabilization and disturbance attenuation of the system in Eqs. (1) with structured uncertainty in Eqs. (2). Let $\delta > 0$ be the desired disturbance-attenuation constant and $P_c > 0$ be a matrix satisfying Eq. (16). With the state-feedback gain $K_c$ in Eq. (7b) obtained using Theorem 1, the closed-loop sytem in Eqs. (15), which can be reduced to Eqs. (9) when $e(t) = 0$, is stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t)$ to $z(t)$ in Eqs. (15) is less than $\delta$ for all $\Delta A$ in Eqs. (2). Now the problem remaining to be solved is to find a robust-$H_\infty$ observer gain $K_o$ in Eq. (7a) for not only reconstructing the state but also achieving both robust stability and $H_\infty$ disturbance attenuation of the closed-loop system in Eqs. (9).

To determine the desired robust-$H_\infty$ observer gain $K_o$, we reformulate the system in Eqs. (9) as

$$\begin{bmatrix} \dot{x}_c(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BK_c & -K_o C \\ \Delta A & A + \Delta A + K_o C \end{bmatrix} \begin{bmatrix} x_c(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} 0 & -K_o S \\ D & K_o S \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \tag{18a}$$

$$z(t) = \begin{bmatrix} C_1 & C_1 \\ D_1 K_c & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ e(t) \end{bmatrix} \tag{18b}$$

When $x_c(t) = 0$, the preceding system can be reduced to

$$\dot{e}(t) = (A + \Delta A + K_o C)e(t) + [D, K_o S]\hat{w}(t) \tag{19a}$$

$$z(t) = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} e(t) \tag{19b}$$

which is dual to the one in Eqs. (15). Hence, the following theorem, which is dual to Theorem 1, is developed to find the observer gain $K_o$ such that the closed-loop system in Eqs. (19) is stable and the $H_\infty$ norm of the transfer function matrix from

$\hat{w}(t)$ to $z(t)$ in Eqs. (19) is less than some prespecified value for all $\Delta A$ in Eqs. (2).

*Theorem 2.* Consider the closed-loop system in Eqs. (19) with the structured uncertainty described in Eqs. (2). Let $\delta > 0$ be a given disturbance-attenuation constant. Suppose that there exist positive scalars $\epsilon_i > 0$, $i = 1, \ldots, \ell$, and $\infty > \hat{\epsilon} > \sigma^2_{\max}(S)/4\delta$ and a matrix $Q_o > 0$ such that the Riccati equation

$$AP_o + P_oA^T + \sum_{i=1}^{\ell}\left(\epsilon_i P_o U_i U_i^T P_o + \frac{1}{\epsilon_i} T_i T_i^T\right) + \frac{\hat{\epsilon}}{\delta} P_o C_1^T C_1 P_o + \frac{1}{\hat{\epsilon}\delta} DD^T - P_o C^T\left(I - \frac{1}{4\hat{\epsilon}\delta} SS^T\right) CP_o + Q_o = 0 \qquad (20)$$

has a SPD solution $P_o$, where $T_i$ and $U_i$, $i = 1, \ldots, \ell$, are defined in Eq. (2c). Then the closed-loop system in Eqs. (19) is stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t)$ to $z(t)$ in Eqs. (19) is less than $\delta$ for all $\Delta A$ in Eqs. (2) with

$$K_o = -\gamma_o P_o C^T \qquad (21a)$$

where $\gamma_o$ satisfies either

$$\frac{2\hat{\epsilon}\delta}{\sigma^2_{\max}(S)} - \frac{1}{2} \geq \gamma_o \geq \frac{1}{2} \quad \text{or} \quad \frac{1}{2} \geq \gamma_o \geq \frac{2\hat{\epsilon}\delta}{\sigma^2_{\min}(S)} - \frac{1}{2} > 0 \qquad (21b)$$

■

Let $\delta > 0$ be the given disturbance-attenuation constant and $P_c > 0$ and $P_o > 0$ be the matrices satisfying Eqs. (16) and (20), respectively. Then the respective $K_c$ and $K_o$ in the observer-based dynamic control law in Eqs. (7) can be obtained using the respective Theorems 1 and 2 for both robust stabilization and $H_\infty$ disturbance attenuation of the system in Eqs. (1) with the structured uncertainty in Eqs. (2). However, the resulting closed-loop system in Eqs. (9) may not be stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t)$ to $z(t)$ in Eqs. (9) may not be less than $\delta$ for all $\Delta A$ in Eqs. (2) due to the failure of the separation theorem for robust stabilization and/or $H_\infty$ control design of uncertain linear systems.

For convenience, we rewrite the augmented system in Eqs. (9) as follows:

$$\dot{\hat{x}}(t) = (\hat{A} + \Delta\hat{A})\hat{x}(t) + \hat{D}\hat{w}(t) \qquad (22a)$$

$$z(t) = \hat{E}\hat{x}(t) \qquad (22b)$$

where $\hat{w}(t)$ is as previously defined, and

$$\hat{x}(t) \triangleq \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad \hat{A} \triangleq \begin{bmatrix} A + BK_c & -BK_c \\ 0 & A + K_oC \end{bmatrix}$$

$$\Delta\hat{A} \triangleq \begin{bmatrix} \Delta A & 0 \\ \Delta A & 0 \end{bmatrix}, \quad \hat{D} \triangleq \begin{bmatrix} D & 0 \\ D & K_oS \end{bmatrix}$$

$$\hat{E} \triangleq \begin{bmatrix} C_1 & 0 \\ D_1K_c & -D_1K_c \end{bmatrix}$$

From Eqs. (2) the structured uncertainty matrix $\Delta\hat{A}$ can be expressed as

$$\Delta\hat{A} = \sum_{i=1}^{\ell} a_i \hat{T}_i \hat{U}_i^T \quad \text{with} \quad \hat{T}_i = \begin{bmatrix} T_i \\ T_i \end{bmatrix} \quad \text{and} \quad \hat{U}_i = \begin{bmatrix} U_i \\ 0 \end{bmatrix} \qquad (23)$$

where $a_i$, $T_i$, and $U_i$, $i = 1, \ldots, \ell$, are defined in Eqs. (2).

The following lemma gives a sufficient condition for both robust stability and $H_\infty$ disturbance attenuation of the closed-loop system in Eqs. (22) with the structured uncertainty in Eq. (23).

*Lemma 2.* Consider the closed-loop system in Eqs. (22) with the structured uncertainty described in Eq. (23). Let $\delta > 0$ be the given disturbance-attenuation constant. Suppose that there exist positive scalars $\zeta_i > 0$, $i = 1, \ldots, \ell$, and $\infty > \hat{\zeta} > 0$, and a matrix $\hat{Q} > 0$ such that the Riccati equation

$$\hat{A}^T\hat{P} + \hat{P}\hat{A} + \sum_{i=1}^{\ell}\left(\zeta_i \hat{P}\hat{T}_i\hat{T}_i^T\hat{P} + \frac{1}{\zeta_i}\hat{U}_i\hat{U}_i^T\right) + \frac{\hat{\zeta}}{\delta}\hat{P}\hat{D}\hat{D}^T\hat{P} + \frac{1}{\hat{\zeta}\delta}\hat{E}^T\hat{E} + \hat{Q} = 0 \qquad (24)$$

has a SPD solution $\hat{P}$, where $\hat{T}_i$ and $\hat{U}_i$, $i = 1, \ldots, \ell$, are defined in Eq. (23). Then the closed-loop system in Eqs. (22) is stable and the $H_\infty$ norm of the transfer function matrix from $\hat{w}(t)$ to $z(t)$ in Eqs. (22) is less than $\delta$ for all $\Delta\hat{A}$ in Eq. (23).

*Proof.* Suppose that $\hat{P} > 0$ satisfies the Riccati equation in Eqs. (24). Using the equality in Eqs. (24) and the matrix inequality in Eq. (10), we obtain the following matrix inequality:

$$-(\hat{A} + \Delta\hat{A})^T\hat{P} - \hat{P}(\hat{A} + \Delta\hat{A}) - \frac{\hat{\zeta}}{\delta}\hat{P}\hat{D}\hat{D}^T\hat{P} - \frac{1}{\hat{\zeta}\delta}\hat{E}^T\hat{E}$$

$$= \sum_{i=1}^{\ell}\left(\zeta_i\hat{P}\hat{T}_i\hat{T}_i^T\hat{P} + \frac{1}{\zeta_i}\hat{U}_i\hat{U}_i^T\right) - \Delta\hat{A}^T\hat{P} - \hat{P}\Delta\hat{A} + \hat{Q}$$

$$\geq \sum_{i=1}^{\ell}|a_i|\left[\zeta_i\hat{P}\hat{T}_i\hat{T}_i^T\hat{P} + \frac{1}{\zeta_i}\hat{U}_i\hat{U}_i^T \pm (\hat{P}\hat{T}_i\hat{U}_i^T + \hat{U}_i\hat{T}_i^T\hat{P})\right] + \hat{Q} \geq \hat{Q} > 0$$

for all $\Delta\hat{A}$ in Eq. (23). Hence, by Lemma 1, we conclude that the closed-loop system in Eqs. (22) is stable and satisfies the $H_\infty$-norm bound $\delta$ for all $\Delta\hat{A}$ in Eq. (23). ■

## Design for a Benchmark Control Problem

To highlight the concepts and methodologies presented in the previous sections, design problems 1 and 3 of the benchmark example[15-18] associated with a mass-spring system, which approximates the dynamics of a flexible structure, are considered here. The two-mass/spring single input/single output system, shown in Fig. 1, is described by

$$\ddot{x}_1(t) + \frac{k}{m_1}[x_1(t) - x_2(t)] = \frac{u(t)}{m_1} \qquad (25a)$$

$$\ddot{x}_2(t) + \frac{k}{m_2}[x_2(t) - x_1(t)] = \frac{w(t)}{m_2} \qquad (25b)$$

with a noncollocated measurement

$$y(t) = x_2(t) + Sv(t) \qquad (25c)$$

and the controlled output

$$z(t) = \begin{bmatrix} x_2(t) \\ D_1u(t) \end{bmatrix} \qquad (25d)$$

where $u(t)$ is an actuator input; $w(t)$ is a disturbance input; $v(t)$ is a white Gaussian noise process with unit power spectral density and $S = 0.01$, $m_1 = m_2 = 1$; $k$ is an unknown but bounded uncertain stiffness parameter with $0.5 \leq k \leq 2$; and the weighting matrix $D_1$ is to be chosen upon design.

Let $k_{\text{nom}}$ and $\Delta k$ denote the nominal value and variation of the uncertain parameter $k = k_{\text{nom}} + \Delta k$, respectively. Then, by defining $x_3(t) = \dot{x}_1(t)$ and $x_4(t) = \dot{x}_2(t)$, we can represent this

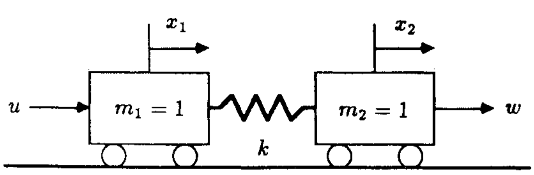


Fig. 1 Two-mass/spring system.

uncertain linear system as that in Eqs. (1) with the nominal system matrices given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_{\text{nom}} & k_{\text{nom}} & 0 & 0 \\ k_{\text{nom}} & -k_{\text{nom}} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = C_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T \tag{26a}$$

and the structured parameter uncertainty given by

$$\Delta A = a_1 A_1, \qquad -1 \leq a_1 \leq 1,$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\Delta\bar{k} & \Delta\bar{k} & 0 & 0 \\ \Delta\bar{k} & -\Delta\bar{k} & 0 & 0 \end{bmatrix} \tag{26b}$$

where $\Delta\bar{k}$ is a scalar such that $\Delta\bar{k} \geq |\Delta k|$. Furthermore, the matrix $A_1$ can be decomposed into $A_1 = T_1 U_1^T$ as that in Eq. (2c) using the SVD technique[19] with

$$T_1 = \begin{bmatrix} 0 & 0 & \sqrt{\Delta\bar{k}} & -\sqrt{\Delta\bar{k}} \end{bmatrix}^T$$

and

$$U_1 = \begin{bmatrix} -\sqrt{\Delta\bar{k}} & \sqrt{\Delta\bar{k}} & 0 & 0 \end{bmatrix}^T \tag{26c}$$

**Design Problem 1**

This problem considers only robust stabilization but not disturbance attenuation (i.e., $\delta \to \infty$) of the uncertain system in Eqs. (25). To find a suitable control law in Eqs. (7) that guarantees the stability of the uncertain system in Eqs. (25) for $0.5 \leq k \leq 2.0$, we let $k_{\text{nom}} = 1.25$ and $\Delta\bar{k} = 0.75$. After trial-and-error iterations with successive reduction of the tuning parameters $\xi_1$, $\epsilon_1$, and $\zeta_1$, a robust controller is obtained as follows. First, a robust state-feedback gain $K_c$ in Eqs. (7) is determined using Theorem 1 as follows. With $Q_c = I$ and $\xi_1 = 0.01$ [the terms $(\hat{\epsilon}/\delta)P_c DD^T P_c$ and $1/(\hat{\epsilon}\delta C_1^T C_1)$ in Eq. (16) vanish for $\delta \to \infty$], the Riccati equation in Eq. (16) has a SPD solution $P_c$. With $\gamma_c = ½$, a robust state-feedback gain can be obtained from Eqs. (17) as

$$K_c = -\gamma_c B^T P_c = [-5.1696 \quad 4.2561 \quad -2.3294 \quad -1.3973] \tag{27}$$

Then Theorem 2 is utilized to find a robust observer gain $K_o$ in Eqs. (7) as follows. With $Q_o = I$ and $\epsilon_1 = 0.005$, the Riccati equation in Eq. (20) has a SPD solution $P_o$. From Eqs. (21), we choose $\gamma_o = ½$ and obtain

$$K_o = -\gamma_o P_o C^T = [-1.5136 \quad -2.6720 \quad 5.9842 \quad -6.8848]^T \tag{28}$$

Combining $K_c$ in Eq. (27) and $K_o$ in Eq. (28) yields the following output-feedback dynamic control law as that in Eqs. (7):

$$\dot{x}_c(t) = (A + BK_c + K_o C)x_c(t) - K_o y(t)$$

$$= \begin{bmatrix} 0 & -1.5136 & 1 & 0 \\ 0 & -2.6720 & 0 & 1 \\ -6.4196 & 11.4903 & -2.3294 & -1.3973 \\ 1.2500 & -8.1348 & 0 & 0 \end{bmatrix} x_c(t)$$

$$+ \begin{bmatrix} 1.5136 \\ 2.6720 \\ -5.9842 \\ 6.8848 \end{bmatrix} y(t) \tag{29a}$$

$$u(t) = K_c x_c(t) = [-5.1696 \quad 4.2561 \quad -2.3294 \quad -1.3973] x_c(t) \tag{29b}$$

By Lemma 2, the dynamic control law in Eqs. (29) guarantees the stability of the uncertain system in Eqs. (25) for all $0.5 \leq k \leq 2.0$, since the associated augmented Riccati equation in Eq. (24) has a SPD solution for $\hat{Q} = I$ and $\zeta_1 = 0.01$. In fact, with the robust dynamic control law given in Eqs. (29), the closed-loop system is found to be stable for all $0.36 \leq k \leq 3.27$. Note that the stability range of the stiffness $k$ has been significantly increased. The dynamic control law (29) is stable but non-minimum phase with poles located at $-0.9988 \pm j3.0265$ and $-1.5019 \pm j1.5378$ and zeros located at $-7.6327$, $-0.0766$, and $0.2238$. The gain and phase margins of the loop transfer function with this robust control law for $k = 1.0$ are 1.197 (1.56 dB) and 16.06 deg, respectively. Such gain and phase margins are relatively small compared to typical 6-dB and 40-deg margins employed in practice. However, it is important to note that the control design problem here is concerned with stability robustness with respect to plant uncertainty caused by parametric variation of the spring stiffness $k$.

The robust control law (29) is somewhat conservative (as can be seen from the enlargement of the stability range) for this particular control problem. After a certain amount of trial and error, a less conservative control law is obtained as follows. We choose $k_{\text{nom}} = 1.0$ and $\Delta\bar{k} = 0.45$. Following the same procedure as that given earlier, a robust state-feedback gain $K_c$ is determined as

$$K_c = [-25.8854 \quad 11.2182 \quad -8.6724 \quad -28.2250] \tag{30}$$

by solving the Riccati equation in Eq. (16) with $Q_c = 20I$, $\epsilon_1 = 0.02$, and $\gamma_c = 1$ in Eqs. (17). Then a robust observer gain $K_o$ is determined as

$$K_o = [-2.0057 \quad -3.0953 \quad 2.5573 \quad -4.2719]^T \tag{31}$$

by solving the Riccati equation in Eq. (20) with $Q_o = I$, $\epsilon_1 = 0.07$, and $\gamma_o = 1$ in Eqs. (21). The resulting output-feedback dynamic control law in Eqs. (7) with $K_c$ in Eq. (30) and $K_o$ in Eq. (31) is

$$\dot{x}_c(t) = \begin{bmatrix} 0 & -2.0057 & 1 & 0 \\ 0 & -3.0953 & 0 & 1 \\ -26.8854 & 14.7756 & -8.6724 & -28.2250 \\ 1 & -5.2719 & 0 & 0 \end{bmatrix} x_c(t)$$

$$+ \begin{bmatrix} 2.0057 \\ 3.0953 \\ -2.5573 \\ 4.2719 \end{bmatrix} y(t) \tag{32a}$$

$$u(t) = [-25.8854 \quad 11.2182 \quad -8.6724 \quad -28.2250] x_c(t) \tag{32b}$$

The stability range of the robust control law (32) is found to be $0.497 \leq k \leq 2.10$. The dynamic control law (32) is stable but non-minimum phase, with poles located at $-4.3594 \pm j1.6653$ and $-1.5245 \pm j2.8838$ and zeros located at $-0.1554$ and $0.6635 \pm j0.9801$. Root locus vs overall loop gain of this con-

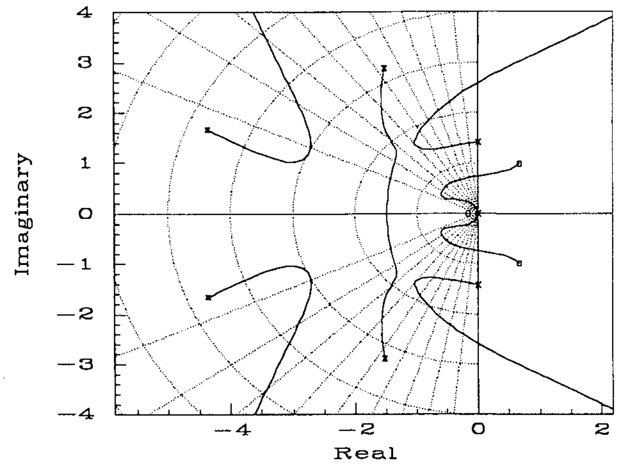


**Fig. 2 Root locus vs overall loop gain with control law in Eqs. (32) for $k = 1.0$.**

trol law for the nominal spring stiffness $k = 1.0$ is shown in Fig. 2. The gain and phase margins of the loop transfer function with the robust control law for $k = 1.0$ are 1.466 (3.32 dB) and 29.61 deg, respectively.

Let $w(t)$ be a unit-impulse disturbance, and let the initial conditions be zero for the following simulations. The time response of $x_1(t)$, $x_2(t)$, and $u(t)$ with the control law in Eqs. (32) for $k = 1.0$ are shown in Fig. 3. It is observed from Fig. 3 that the peak magnitude of the controlled variable $x_2(t)$ is $\sim 1.5$ and that $x_2(t)$ has settled down in $< 11.0$ s for $k = 1.0$. Note that the control law (29) achieves a larger stability range $(0.36 \leq k \leq 3.27)$ than the control law (32) $(0.497 \leq k \leq 2.10)$, at the expense of less satisfactory time responses [with a peak magnitude of $x_2(t)$ at around 5.0 and a settling time around 19.0 s for $k = 1.0$] due to its conservativeness.

**Design Problem 3**

This problem considers both robust stabilization and disturbance attenuation of the uncertain linear system in Eqs. (25). In particular, we consider the case that the desired value of $H_\infty$ disturbance-attenuation bound $\delta$ is unity. We also consider the case that a designed observer-based robust controller will reject the cyclic disturbance $w(t)$, which is caused by a sinusoidal disturbance of frequency 0.5 rad/s with unknown but constant amplitude and phase.

*Case 1*

We set the desired disturbance-attenuation constant $\delta = 1$ and let the weighting matrix $D_1 = 0.005$. We let $k_{\mathrm{nom}} = 1.0$ and $\Delta \bar{k} = 0.5$. First, a robust-$H_\infty$ state-feedback gain is determined as

$$K_c = [-338.559 \quad -1512.29 \quad -46.3161 \quad -1271.49] \tag{33}$$

by solving the Riccati equation in Eq. (16) with $Q_c = 0$, $\xi_1 = 5.0e-4$, $\hat{\xi} = 1.25e-5$, and $\gamma_c = 1$ in Eqs. (17). Then a robust-$H_\infty$ observer gain is determined as

$$K_o = [-303.608 \quad -33.6515 \quad 73.3764 \quad -282.167]^T \tag{34}$$

by solving the Riccati equation in Eq. (20) with $Q_o = 0$, $\epsilon_1 = 5.0e-5$, $\hat{\epsilon} = 5.0e-5$, and $\gamma_o = 1$ in Eqs. (21). The resulting output-feedback dynamic control law in Eqs. (7) with $K_c$ in Eq. (33) and $K_o$ in Eq. (34) is

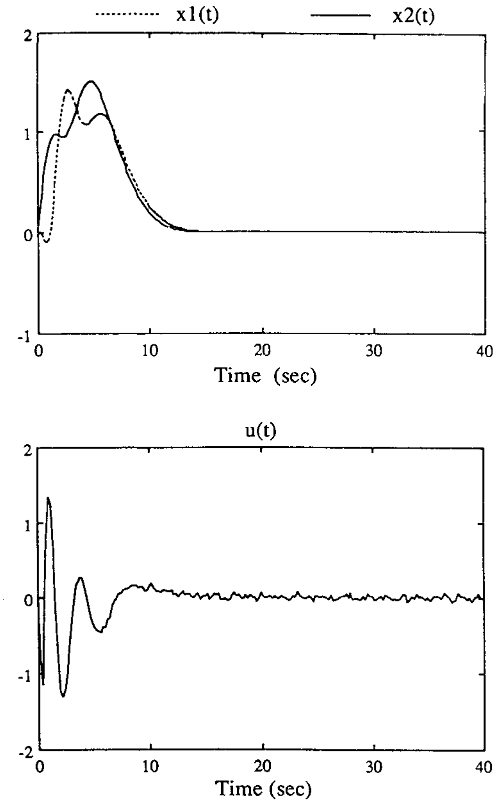


**Fig. 3 Time response to a unit impulse disturbance with control law in Eqs. (32) for $k = 1.0$.**

The poles of this dynamic control law are $-37.2122$, $-26.3170$, and $-8.2191 \pm j9.4946$, and the zeros are $-0.5995$ and $-0.4198 \pm j1.0425$. Notice that the dynamic control law is minimum phase and stabilizes the uncertain system in Eqs. (25) for $0.382 \leq k \leq 2.32$. The gain and phase margins of the loop transfer function with the robust control law for $k = 1.0$ are 2.363 (7.47 dB) and 30.23 deg, respectively.

For the following simulations we let $w(t) = \sin 0.5t$, and we let initial conditions be zero. The time responses of $x_1(t)$, $x_2(t)$, and $u(t)$ with the control law in Eqs. (35) for $k = 1.0$ are shown in Fig. 4. It is seen that $x_2(t)$ satisfies the $H_\infty$ disturbance-attenuation bound $\delta = 1.0$ with a peak magnitude of $\sim 0.3$ and that $x_2(t)$ has settled down $\sim 20$ s for $k = 1.0$. In fact, the actual $H_\infty$-norm of the closed-loop transfer function for $k = 1.0$ is found to be 0.405. Hence, the control law in Eqs. (35) achieves good disturbance attenuation in $x_2(t)$ against $w(t)$ of any frequency. It is noted that the control effort $u(t)$ is quite large due to the measurement sensor noise.

*Case 2*

Let $w(t)$ be a cyclic disturbance described by

$$w(t) = A_w \sin(0.5t + \phi)$$

where $A_w$ and $\phi$ are unknown but constant. Since the frequency of the disturbance is 0.5 rad/s, we can differentiate

$$\dot{x}_c(t) = \begin{bmatrix} 0 & -303.608 & 1 & 0 \\ 0 & -33.6515 & 0 & 1 \\ -339.559 & -1437.92 & -46.3161 & -1271.29 \\ 1 & -283.167 & 0 & 0 \end{bmatrix} x_c(t) + \begin{bmatrix} 303.608 \\ 33.6515 \\ -73.3764 \\ 282.167 \end{bmatrix} y(t) \tag{35a}$$

$$u(t) = [-338.559 \quad -1512.29 \quad -46.3161 \quad -1271.49] x_c(t) \tag{35b}$$

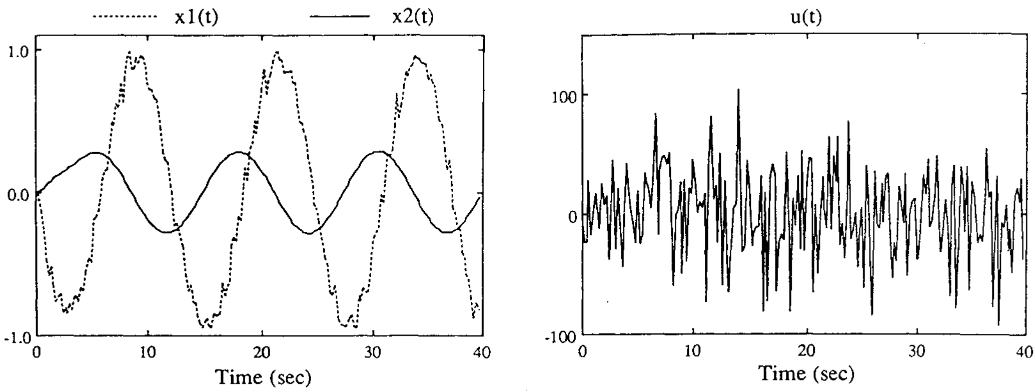


**Fig. 4 Time responses to a sinusoidal disturbance with control law in Eqs. (35) for $k = 1.0$.**

Eqs. (25a) and (25b) twice until $w(t)$ disappears in the resulting system[18] and obtain

$$x_1^{(4)}(t) = -k\left[\ddot{x}_1(t)+0.25x_1(t)-\ddot{x}_2(t)-0.25x_2(t)\right] - 0.25\ddot{x}_1(t) + \hat{u}(t) \tag{36a}$$

$$x_2^{(4)}(t) = -k\left[\ddot{x}_2(t)+0.25x_2(t)-\ddot{x}_1(t)-0.25x_1(t)\right] - 0.25\ddot{x}_2(t) \tag{36b}$$

where $\hat{u}(t)$ is a new control variable defined as

$$\hat{u}(t) \triangleq \ddot{u}(t) + 0.25u(t) \tag{37}$$

The new system (36) contains uncontrollable poles at $s = \pm j0.5$. Hence, a new state, $\hat{x}_1(t) \triangleq \ddot{x}_1(t)+0.25x_1(t)$, is introduced to remove the uncontrollable poles from Eqs. (36). Then Eqs. (36) become

$$\ddot{\hat{x}}_1(t) = -k\hat{x}_1(t) + k\left[\ddot{x}_2(t)+0.25x_2(t)\right] + \hat{u}(t) \tag{38a}$$

$$x_2^{(4)}(t) = -(k+0.25)\ddot{x}_2(t) - 0.25kx_2(t) + k\hat{x}_1(t) \tag{38b}$$

For the uncertain system given in Eqs. (38), we assume that $k_{\text{nom}} = 1.0$ and $\Delta\bar{k} = 0.75$. Define

$$\hat{x}(t) = \left[\hat{x}_1(t), \dot{\hat{x}}_1(t), x_2(t), \dot{x}_2(t), \ddot{x}_2(t), x_2^{(3)}(t)\right]^T$$

Then Eqs. (25c) and (38) can be represented as

$$\dot{\hat{x}}(t) = (\hat{A} + \Delta\hat{A})\hat{x}(t) + \hat{B}\hat{u}(t) \tag{39a}$$

$$y(t) = \hat{C}\hat{x}(t) + Sv(t) \tag{39b}$$

where the nominal system matrices are given by

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0.25 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -0.25 & 0 & -1.25 & 0 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

and the structured parameter uncertainty is given by

$$\hat{A} = \hat{a}_1\hat{A}_1, \qquad -1 \le \hat{a}_1 \le 1$$

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.75 & 0 & 0.1875 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.75 & 0 & -0.1875 & 0 & -0.75 & 0 \end{bmatrix}$$

The matrix $\hat{A}_1$ can be decomposed into $\hat{A}_1 = \hat{T}_1\hat{U}_1^T$ using the SVD technique[19] with

$$\hat{T}_1 = [0 \quad 0.66081 \quad 0 \quad 0 \quad 0 \quad -0.66081]^T$$

and

$$\hat{U}_1 = [0 \quad -0.65072 \quad 0 \quad 0.16268 \quad 0 \quad 0.65072]^T$$

To find a suitable control law of the form

$$\dot{\hat{x}}_c(t) = (\hat{A} + \hat{B}\hat{K}_c + \hat{K}_o\hat{C})\hat{x}_c(t) - \hat{K}_o y(t) \tag{40a}$$

$$\hat{u}(t) = \hat{K}_c\hat{x}_c(t) \tag{40b}$$

as in Eqs. (7) to stabilize the uncertain system in Eqs. (39), we follow the same procedure as in design problem 1. First, a robust state-feedback gain is determined as

$$\hat{K}_c = [-149.934 \quad -20.1325 \quad 16.9424 \quad -99.5035 \quad -17.9660 \quad -319.192] \tag{41}$$

by solving the Riccati equation in Eq. (16) with $Q_c = 100I$, $\xi_1 = 8.0e-5$, and $\gamma_c = 1$ in Eqs. (17). Then a robust observer gain is determined as

$$\hat{K}_o = [-3.68632 \quad 67.4489 \quad 15.1218 \quad -63.1613 \quad -125.940 \quad -77.3212]^T \tag{42}$$

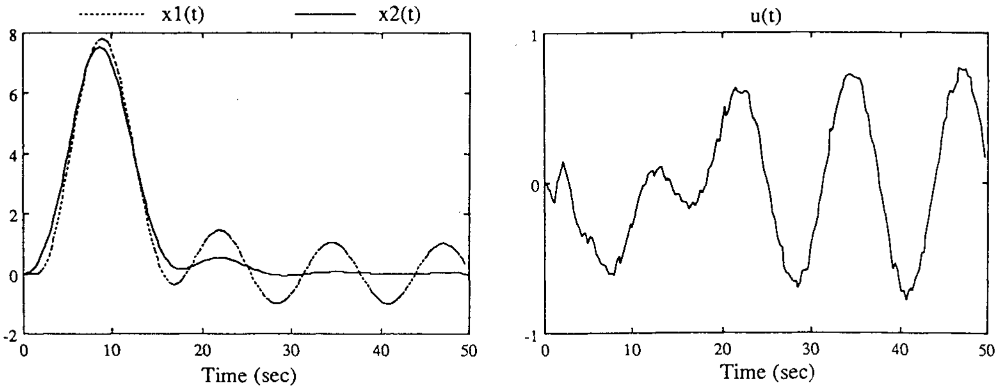


**Fig. 5 Time response to a sinusoidal disturbance with control law in Eqs. (44) for $k = 1.0$.**

by solving the Riccati equation in Eq. (20) with $Q_o = 100I$, $\epsilon_1 = 2.0e-4$, and $\gamma_o = 1$ in Eqs. (21). The resulting output-feedback dynamic control law in Eqs. (40) with $\hat{K}_c$ in Eq. (41) and $\hat{K}_o$ in Eq. (42) becomes

$$\dot{\hat{x}}_c(t) = \hat{A}_c\hat{x}_c(t) + \hat{B}_c y(t), \qquad \hat{u}(t) = \hat{C}_c y(t) \tag{43}$$

where

$$\hat{A}_c = \begin{bmatrix} 0 & 1 & -3.68632 & 0 & 0 & 0 \\ -150.934 & -20.1325 & 84.6414 & -99.5035 & -16.9660 & -319.192 \\ 0 & 0 & -15.1218 & 1 & 0 & 0 \\ 0 & 0 & -63.1613 & 0 & 1 & 0 \\ 0 & 0 & -125.940 & 0 & 0 & 1 \\ 1 & 0 & -77.5712 & 0 & -1.2500 & 0 \end{bmatrix}$$

$$\hat{B}_c = [3.68632 \quad -67.4489 \quad 15.1218 \quad 63.1613 \quad 125.940 \quad 77.3212]^T$$

$$\hat{C}_c = [-149.934 \quad -20.1325 \quad 16.9424 \quad -99.5035 \quad -17.9660 \quad -319.192]$$

Combining Eqs. (37) and (43) yields the following eighth-order dynamic output-feedback control law:

$$\dot{x}_c(t) = \begin{bmatrix} \hat{A}_c & 0_{6\times 2} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}\hat{C}_c & \begin{pmatrix} 0 & 1 \\ -0.25 & 0 \end{pmatrix} \end{bmatrix} x_c(t) + \begin{bmatrix} \hat{B}_c \\ 0 \end{bmatrix} y(t) \tag{44a}$$

$$u(t) = [0_{1\times 6} \quad 1 \quad 0]x_c(t) \tag{44b}$$

The poles of the dynamic control law are $\pm j0.5$, $-9.50$, $-6.3214$, $-8.2575 \pm j4.8882$, and $-1.4589 \pm j3.1428$, and the zeros are $-0.0821$, $0.6762 \pm j0.7221$, and $0.0645 \pm j0.4498$. Notice that the dynamic control law in Eqs. (44) has four non-minimum phase zeros and stabilizes the uncertain system in Eqs. (25) for $0.498 \le k \le 2.02$. The gain and phase margins of the loop transfer function with the robust control law for $k = 1.0$ are 1.659 (4.40 dB) and 33.99 deg, respectively.

For the following simulations, again, we let $w(t) = \sin 0.5t$, and we let initial conditions be zero. The time responses of $x_1(t)$, $x_2(t)$, and $u(t)$ with the control law in Eqs. (44) for $k = 1.0$ are shown in Fig. 5. The peak magnitude of the controlled variable $x_2(t)$ for $k = 1.0$ is $\sim 7.6$. Also, Fig. 5 shows that, for $k = 1.0$, $x_2(t)$ has settled down and the cyclic disturbance is rejected in $x_2(t)$ within 20 s.

*Remark 4.* Based on our experience in numerous computer simulations for finding a suitable tradeoff among system performance, robust stability, and disturbance attenuation, we have the following observations: The weighting matrices $Q_c$ and $Q_o$ are important factors for achieving the requirements of system performance; the tuning parameters $\xi_i$ and $\epsilon_i$ play dominant roles for robust stability; and the value of $\delta$ has a significant effect on disturbance attenuation of the closed-loop system. In all, the choice of the values of the introduced parameters is a design freedom, and a certain amount of experience is helpful. ■

## Conclusions

Based on the algebraic Riccati equation approach and Lyapunov stability theory, a new observer-based robust-$H_\infty$ output-feedback control law has been developed for both robust stabilization and disturbance attenuation with $H_\infty$-norm bound for an uncertain linear system. These observer-based disturbance-attenuation robust-stabilizing control laws can be easily constructed from the symmetric positive-definite solution of a pair of augmented Riccati equations. A simple dual concept has been utilized for finding the robust-$H_\infty$ state-feedback gain $K_c$ and the robust-$H_\infty$ observer gain $K_o$. Also, the proposed approach is more flexible than some existing methods in the sense that additional tuning parameters (e.g., $\xi$, $\epsilon$, and $\zeta$) have been introduced in the derivations to achieve robust stabilization and disturbance attenuation for uncertain linear systems. A benchmark problem associated with a mass-spring system has been used to illustrate the design methodologies.

## Acknowledgments

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